Integration by Parts

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We have seen that by the Fundamental Theorem of Calculus, we can evaluate a given integral if we can find an antiderivative of the integrand function. But in general, it is not easy to find antiderivatives directly, so we need more techniques.

Theorem 1. *(Integration by Parts)*

$$\int f(x)g'(x) = f(x)g(x) - \int g(x)f'(x)\,dx$$

Proof. By the Leibniz Rule of differentiating a product of functions, we know

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

So $f(x)g(x)$ is an antiderivative of $f'(x)g(x) + f(x)g'(x)$,

$$\int f'(x)g(x) + f(x)g'(x)\,dx = f(x)g(x) + C$$

We then see

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

□

Remark 2. Recall that given a differentiable function f, there is a corresponding differential $df = f'(x)dx$, so the aove theorem can also be written as

$$\int f(x)\,dg(x) = f(x)g(x) - \int g(x)\,df(x)$$

We can use the above theorem to find antiderivatives of product of functions.

Example 3. *Find antiderivatives of $f(x) = xe^x$.*

$$\begin{aligned}\int xe^x\,dx &= \int x(e^x)'\,dx \\ &= xe^x - \int (x)'e^x\,dx \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C\end{aligned}$$

If you like to use the differential notation instead, you will get

$$\begin{aligned}\int xe^x\,dx &= \int x\,de^x \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C\end{aligned}$$

Example 4. *Compute $\int \frac{1}{x}\ln x\,dx$*

$$\begin{aligned}\int \frac{1}{x}\ln x\,dx &= \int (\ln x)'\ln x\,dx \\ &= (\ln x)(\ln x) - \int \ln x(\ln x)'\,dx \\ &= (\ln x)^2 - \int \frac{1}{x}\ln x\,dx\end{aligned}$$

We see $\int \frac{1}{x}\ln x\,dx = \frac{1}{2}(\ln x)^2 + C$

We can also apply the method of Integration by Parts in evaluating definite integrals:

Theorem 5.

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx$$

Proof. We have seen that $f(x)g(x)$ is an antiderivative of $f(x)g'(x)+f'(x)g(x)$. Then by the Fundamental Theorem of Calculus,

$$\int_a^b f(x)g'(x) + f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a)$$

$\square$

Example 6. *Evaluate* $\int_1^3 x \ln x\,dx$

$$\begin{aligned}
\int_1^3 x\ln x\,dx &= \int_1^3 \ln x d\frac{x^2}{2} \\
&= \frac{x^2}{2}\ln x\Big|_1^3 - \int_1^3 \frac{x^2}{2}\,d\ln x \\
&= \frac{9}{2}\ln 3 - \int_1^3 \frac{x^2}{2}\frac{1}{x}\,dx \\
&= \frac{9}{2}\ln 3 - \int_1^3 \frac{x}{2}\,dx \\
&= \frac{9}{2}\ln 3 - \frac{x^2}{4}\Big|_1^3 \\
&= \frac{9}{2}\ln 3 - 2
\end{aligned}$$